Robustness of the nonlinear PI control method to ignored actuator dynamics by Haris E. Psillakis [1]

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***Abstract*— For sector bounded nonlinear systems with unknown control direction, a nonlinear PI or Nussbaum gain controller is not sufficient for stability. This paper shows that combining the nonlinear PI and Nussbaum gain will result in a stable system.**

I. Introduction

This paper explores a control problem where the sign of the control, or control direction, is unknown. The common way to handle such a problem is through the use of Nussbaum functions as control gains. Nussbaum functions are defined as continuous functions $N : \mathbb{R} \to \mathbb{R}$ for which the properties of (1) and (2) hold.

$$\limsup_{\zeta\to\pm\infty} \frac{1}{\zeta}\int_0^{\zeta} N(s)\,ds = +\infty \tag{1}$$

$$\liminf_{\zeta\to\pm\infty} \frac{1}{\zeta}\int_0^{\zeta} N(s)\,ds = -\infty \tag{2}$$

The author has previously shown that combining a nonlinear PI controller with a Nussbaum gain can stabilize a perturbed linear system such as (3), by using the control scheme represented in (4) and (5) as long as $\max\left\{\epsilon\lambda, \epsilon(\alpha+\lambda)\right\} < 1$ and $\kappa(\cdot)$ is a Nussbaum function.

$$\left\{ \begin{array}{c} \dot{x} = \alpha x + bu \\ \varepsilon\dot{y} = x - y \end{array} \right\} \tag{3}$$

$$u = \kappa(z)\,y \tag{4}$$

$$z = \frac{1}{2}y^2 + \lambda\int_0^t y^2(s)\,ds \tag{5}$$

This paper extends the results of this work to explore the robustness of the controller to ignored actuator dynamics as seen in Figure 2 of [1]. Such a system is modeled in (6).

$$\left\{ \begin{array}{c} \dot{y} = f(y) + bu \\ \varepsilon\dot{u} = u_{nom} - u \end{array} \right\} \tag{6}$$

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where u_{nom} is modeled for the unperturbed plant seen in (7).

$$\dot{y} = f(y) + bu_{nom} \tag{7}$$

In the following section, it will be shown that combining nonlinear PI with a Nussbaum control gain, results in a controller that is more robust than either technique by itself.

A. Nonlinear PI control: nominal case

Let the sector bounded non-linearity $f(y)$ be defined as follows.

$$f(y) = \alpha(y)\,y \tag{8}$$

$$\alpha_1 \leq \alpha(y) \leq \alpha_1 \quad \forall y \in R \tag{9}$$

Lemma 1 Let the system be (7) with nonlinearity (8), (9).

$$\dot{y} = f(y) + bu_{nom}$$

$$f(y) = \alpha(y)\,y \quad \alpha_1 \leq \alpha(y) \leq \alpha_1 \quad \forall y \in R$$

Consider also the nonlinear PI controller of the form

$$u_{nom} = \kappa(z)\,y \tag{10}$$

$$z = \frac{1}{2}y^2 + \lambda\int_0^t y^2(s)\,ds \tag{11}$$

$(\lambda > 0)$ with PI gain $\kappa(z) \equiv \beta(z)\cos(z)$ and $\beta(\cdot)$ a class $\mathcal{K}_\infty$ function. Then, for the closed loop system we have that z, y, u_{nom} are bounded and $\lim_{x\to\infty} y(t) = \lim_{t\to\infty} u_{nom}(t) = 0$. A function $\beta(\cdot)$ belongs to class $\mathcal{K}_\infty$ if it is continuous, strictly increasing with $\beta(0) = 0$ and $\lim_{x\to+\infty} \beta(z) = +\infty$.
Proof: The proof is given in section 1.1 of [2]. ■

II. Nonlinear PI control: ignored actuator dynamics case

Theorem 1 Let the closed-loop system be given by (6), (10), (11) with sector-bounded nonlinearity given by (8), (9).

$$\left\{ \begin{array}{c} \dot{y} = f(y) + bu \\ \varepsilon\dot{u} = u_{nom} - u \end{array} \right\} \quad u_{nom} = \kappa(z)\,y$$

$$z = \frac{1}{2}y^2 + \lambda\int_0^t y^2(s)\,ds$$

$$f(y) = \alpha(y)\,y \quad \alpha_1 \leq \alpha(y) \leq \alpha_1 \quad \forall y \in R$$

If,

1) $\varepsilon(\lambda+\alpha_2)<1$
2) $\kappa(z)=\beta(z)\cos(z)$ with $\beta(\cdot)$ a $\mathcal{K}_\infty$ function having the property

$$\lim_{z\to+\infty}\left[\frac{\beta(z+\varepsilon)}{z}-c\beta(z)\right]=+\infty \tag{12}$$

then all closed loop signal are bounded and

$$\lim_{t\to\infty}y(t)=\lim_{t\to\infty}u(t)=\lim_{t\to\infty}u_{nom}(t)=0.$$

Proof:
We begin with (5) and (7) (repeated below).

$$z=\tfrac{1}{2}y^2+\lambda\int_0^t y^2(s)\,ds \quad \dot{y}=f(y)+bu$$

We now differentiate z, substituting the expression for $\dot{y}$.

$$\dot{z}=y\dot{y}+\lambda y^2=y\big(f(y)+bu\big)+\lambda y^2$$

We then factor common terms.

$$\dot{z}=byu+\big(f(y)+\lambda y\big)y$$

We now use the expression for $f(y)$.

$$f(y)=\alpha(y)\,y$$

Substituting into the previous equation we arrive at $\dot{z}$.

$$\dot{z}=byu+\big(\alpha(y)+\lambda\big)y^2 \tag{13}$$

We now define the function S to be the following.

$$S\equiv\frac{\varepsilon}{2}u^2+\frac{\varepsilon(\alpha_2+\lambda)}{b}uy+\frac{\ell}{2}y^2 \tag{14}$$

We then compute $\dot{S}$ as follows, first using the product rule.

$$\dot{S}=\varepsilon u\dot{u}+\frac{\varepsilon(\alpha_2+\lambda)}{b}\left[\dot{u}y+\dot{y}u\right]+\ell y\dot{y}$$

We now substitute the expressions for $\dot{u},\dot{y}$ from (6).

$$\begin{aligned}&\dot{S}=\varepsilon u\tfrac{u_{nom}-u}{\varepsilon}\\&+\tfrac{\varepsilon(\alpha_2+\lambda)}{b}\left[\tfrac{u_{nom}-u}{\varepsilon}y+\big(f(y)+bu\big)u\right]\\&+\ell y\big(f(y)+bu\big)\end{aligned}$$

We factor in ε to cancel common factors.

$$\begin{aligned}&\dot{S}=u(u_{nom}-u)\\&+\tfrac{(\alpha_2+\lambda)}{b}\left[(u_{nom}-u)y+\varepsilon\big(f(y)+bu\big)u\right]\\&+\ell y\big(f(y)+bu\big)\end{aligned}$$

We substitute the expression for u_{nom} from (10).

$$\begin{aligned}&\dot{S}=u\big(\kappa(z)y-u\big)\\&+\tfrac{(\alpha_2+\lambda)}{b}\left[\big(\kappa(z)y-u\big)y+\varepsilon\big(\alpha(y)y+bu\big)u\right]\\&+\ell y\big(\alpha(y)y+bu\big)\end{aligned}$$

We then expand terms.

$$\begin{aligned}&\dot{S}=\kappa(z)uy-u^2\\&+\tfrac{(\alpha_2+\lambda)}{b}\left[\kappa(z)y^2-uy+\varepsilon\alpha(y)uy+\varepsilon bu^2\right]\\&+\ell y\big(\alpha(y)y+bu\big)\end{aligned}$$

We group common factors of u^2 and uy.

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2+\kappa(z)uy\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy\\&+\tfrac{1}{b}(\alpha_2+\lambda)\kappa(z)y^2+\ell\alpha(y)y^2+b\ell uy\end{aligned}$$

We then add and subtract $\ell\dot{z}$ using (13).

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2+\kappa(z)uy\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy\\&+\ell\dot{z}-\ell\left[byu+\big(\alpha(y)+\lambda\big)y^2\right]\\&+\tfrac{1}{b}(\alpha_2+\lambda)\kappa(z)y^2+\ell\alpha(y)y^2+b\ell uy\end{aligned}$$

We now cancel common terms.

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2+\kappa(z)uy\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy\\&+\ell\dot{z}-\ell\lambda y^2+\tfrac{1}{b}(\alpha_2+\lambda)\kappa(z)y^2\end{aligned}$$

Rearranging (13), we have an expression for λy^2.

$$\dot{z}=byu+\big(\alpha(y)+\lambda\big)y^2\Rightarrow\lambda y^2=\dot{z}-byu-\alpha(y)y^2$$

We substitute this expression in to arrive at the following.

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2+\kappa(z)uy\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy+\ell\dot{z}-\ell\lambda y^2\\&+\tfrac{1}{b}\kappa(z)\alpha_2y^2+\tfrac{1}{b}\kappa(z)\left[\dot{z}-byu-\alpha(y)y^2\right]\end{aligned}$$

Simplifying, we arrive at $\dot{S}$.

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy-\lambda\ell y^2\\&+\tfrac{1}{b}\big(\alpha_2-\alpha(y)\big)\kappa(z)y^2+\ell\dot{z}+\tfrac{1}{b}\kappa(z)\dot{z}\end{aligned} \tag{15}$$

The above equation for $\dot{S}$ may be written as a linear equation.

$$\begin{aligned}&\tfrac{d}{dt}\left[S-\tfrac{1}{b}\int_0^{z(t)}\big(\kappa(s)+b\ell\big)ds\right]\\&=-w^T\Lambda(y)w+\tfrac{1}{b}\big(\alpha_2-\alpha(y)\big)\kappa(z)y^2\end{aligned} \tag{16}$$

Here, w is the state vector of (u,y).

$$w=\begin{bmatrix}u & y\end{bmatrix}^T$$

$\Lambda(y)$ is defined to be the following matrix.

$$\Lambda(y)=\begin{bmatrix}1-\varepsilon(\lambda+\alpha_2) & \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b}\\ * & \lambda\ell\end{bmatrix} \tag{17}$$

Where * denotes that the matrix is symmetric w.r.t. the main diagonal.
We show the linear equation is correct by expanding the first term.

$$-w^T\Lambda(y)w=-\begin{bmatrix}u & y\end{bmatrix}\cdot\begin{bmatrix}1-\varepsilon(\lambda+\alpha_2) & \frac{1}{2b}(\lambda+\alpha_2)\big(1-\varepsilon\alpha(y)\big)\\ * & \lambda\ell\end{bmatrix}\cdot\begin{bmatrix}u\\ y\end{bmatrix}$$

We compute the first product.

$$-w^T\Lambda(y)w=\begin{bmatrix}-u\big(1-\varepsilon(\lambda+\alpha_2)\big)-y\frac{1}{2b}(\lambda+\alpha_2)\big(1-\varepsilon\alpha(y)\big)\\ -u\frac{1}{2b}(\lambda+\alpha_2)\big(1-\varepsilon\alpha(y)\big)-y\lambda\ell\end{bmatrix}^T\cdot\begin{bmatrix}u\\ y\end{bmatrix}$$

We now compute the second.

$$
\begin{aligned}
&-w^T \Lambda(y) w = \\
&-u^2 \left(1-\varepsilon(\lambda+\alpha_2)\right) - uy\tfrac{1}{2b}(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right) \\
&-uy\tfrac{1}{2b}(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right) - y^2\lambda\ell
\end{aligned}
$$

Simplifying, we arrive at the following.

$$
\begin{aligned}
&-w^T \Lambda(y) w = \\
&-u^2 \left(1-\varepsilon(\lambda+\alpha_2)\right) - uy\tfrac{1}{b}(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right) - y^2\lambda\ell
\end{aligned}
$$

Adding terms, we arrive at the desired expression.

$$
\begin{aligned}
&\tfrac{d}{dt}\left[S(u,y) - \tfrac{1}{b}\int\limits_0^{z(t)} \left(\kappa(s)+b\ell\right) ds\right] \\
&= -w^T\Lambda(y) w + \tfrac{1}{b}\left(\alpha_2 - \alpha(y)\right)\kappa(z) y^2
\end{aligned}
$$

We want to constrain ℓ such that $\Lambda(y)$ is positive definite.

$$
\begin{aligned}
&\Lambda(y) = \\
&\begin{bmatrix} 1-\varepsilon(\lambda+\alpha_2) & \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b} \\ \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b} & \lambda\ell \end{bmatrix} > 0
\end{aligned}
$$

For this to be true the first principle minor must be positive. This yields the following constraint.

$$
1-\varepsilon(\lambda+\alpha_2) > 0 \Rightarrow \varepsilon(\lambda+\alpha_2) < 1
$$

The second principle minor, computed via the determinant, must also be positive.

$$
\begin{vmatrix} 1-\varepsilon(\lambda+\alpha_2) & \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b} \\ \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b} & \lambda\ell \end{vmatrix} > 0
$$

The determinant is computed below.

$$
\left(1-\varepsilon(\lambda+\alpha_2)\right)(\lambda\ell) - \frac{1}{4b^2}(\lambda+\alpha_2)^2\left(1-\varepsilon\alpha(y)\right)^2 > 0
$$

We take the second term to the right hand side.

$$
\left(1-\varepsilon(\lambda+\alpha_2)\right)(\lambda\ell) > \frac{1}{4b^2}(\lambda+\alpha_2)^2\left(1-\varepsilon\alpha(y)\right)^2
$$

We arrive at one constraint on ℓ below.

$$
\ell > \left(\frac{\lambda+\alpha_2}{b}\right)^2 \frac{\left(1-\varepsilon\alpha(y)\right)^2}{4\lambda\left(1-\varepsilon(\lambda+\alpha_2)\right)}
$$

We now write S in terms of a linear equation using $\Lambda'(y)$.

$$
\Lambda'(y) = \begin{bmatrix} \frac{\varepsilon}{2} & \frac{1}{2b}\varepsilon(\lambda+\alpha_2) \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) & \frac{\ell}{2} \end{bmatrix}
$$

We then test this is valid by expanding the following.

$$
\begin{aligned}
&S = w^T\Lambda'(y) w \\
&= \begin{bmatrix} u & y \end{bmatrix} \cdot \begin{bmatrix} \frac{\varepsilon}{2} & \frac{1}{2b}\varepsilon(\lambda+\alpha_2) \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) & \frac{\ell}{2} \end{bmatrix} \cdot \begin{bmatrix} u \\ y \end{bmatrix}
\end{aligned}
$$

We compute the first product.

$$
w^T\Lambda'(y) w = \begin{bmatrix} \frac{\varepsilon}{2}u + \frac{1}{2b}\varepsilon(\lambda+\alpha_2) y \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) u + \frac{\ell}{2}y \end{bmatrix}^T \cdot \begin{bmatrix} u \\ y \end{bmatrix}
$$

Then compute the second product.

$$
\begin{aligned}
&w^T\Lambda'(y) w \\
&= \tfrac{\varepsilon}{2}u^2 + \tfrac{1}{2b}\varepsilon(\lambda+\alpha_2) uy + \tfrac{1}{2b}\varepsilon(\lambda+\alpha_2) uy + \tfrac{\ell}{2}y^2
\end{aligned}
$$

We finally arrive at the desired expression.

$$
w^T\Lambda'(y) w = \frac{\varepsilon}{2}u^2 + \frac{1}{b}\varepsilon(\lambda+\alpha_2) uy + \frac{\ell}{2}y^2
$$

For S to be positive definite, $\Lambda'(y)$ must be also.

$$
\begin{bmatrix} \frac{\varepsilon}{2} & \frac{1}{2b}\varepsilon(\lambda+\alpha_2) \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) & \frac{\ell}{2} \end{bmatrix} > 0
$$

Since the first principle minor is positive by definition, we examine the second.

$$
\begin{vmatrix} \frac{\varepsilon}{2} & \frac{1}{2b}\varepsilon(\lambda+\alpha_2) \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) & \frac{\ell}{2} \end{vmatrix} > 0
$$

We then compute the determinant.

$$
\frac{\varepsilon\ell}{4} - \frac{1}{4b^2}\varepsilon^2(\lambda+\alpha_2)^2 > 0 \Rightarrow \frac{\varepsilon\ell}{4} > \frac{1}{4b^2}\varepsilon^2(\lambda+\alpha_2)^2
$$

We arrive at a second condition on ℓ.

$$
\ell > \frac{1}{b^2}(\lambda+\alpha_2)^2\varepsilon
$$

We now have two constraints on ℓ, both upper bounds.

$$
\ell > \tfrac{1}{b^2}(\lambda+\alpha_2)^2\varepsilon \quad \ell > \left(\tfrac{\lambda+\alpha_2}{b}\right)^2 \tfrac{\left(1-\varepsilon\alpha(y)\right)^2}{4\lambda\left(1-\varepsilon(\lambda+\alpha_2)\right)}
$$

Therefore, ℓ must be greater than their maximum.

$$
\ell > \left(\frac{\lambda+\alpha_2}{b}\right)^2 \max\left\{\varepsilon, \frac{\left(1-\varepsilon\alpha(y)\right)^2}{4\lambda\left(1-\varepsilon(\lambda+\alpha_2)\right)}\right\} \tag{18}
$$

We now integrate (16) to remove the derivative.

$$
\begin{aligned}
&S - S(0) - \tfrac{1}{b}\int\limits_0^{z(t)} \left(\kappa(s)+b\ell\right) ds \\
&= \int\limits_0^t \begin{bmatrix} -w^T(a)\Lambda(y) w(a) \\ +\frac{1}{b}\left(\alpha_2-\alpha(y)\right)\kappa\left(z(s)\right) y^2(s) \end{bmatrix} ds
\end{aligned}
$$

We break up terms.

$$
\begin{aligned}
&S - S(0) - \tfrac{1}{b}\int\limits_0^{z(t)} \left(\kappa(s)\right) ds - \ell z(t) \\
&= -\int\limits_0^t w^T(s)\Lambda(y) w(s) ds \\
&+\tfrac{1}{b}\int\limits_0^t \left(\alpha_2-\alpha(y)\right)\kappa\left(z(s)\right) y^2(s) ds
\end{aligned}
$$

Rearranging terms, we have the following.

$$
\begin{aligned}
&S - \int\limits_0^t w^T(s)\Lambda(y) w(s) ds \\
&= S(0) + \ell z(t) + \tfrac{1}{b}\int\limits_0^{z(t)} \left(\kappa(s)\right) ds \\
&+\tfrac{1}{b}\int\limits_0^t \left(\alpha_2-\alpha(y)\right)\kappa\left(z(s)\right) y^2(s) ds
\end{aligned}
$$

We then rewrite this equation as an inequality.

$$\begin{aligned} &S-\int_0^t w^T(s)\,\Lambda(y)\,w(s)\,ds \\ &\leq S(0)+\ell z(t)+\tfrac{1}{b}\int_0^{z(t)}\big(\kappa(s)\big)\,ds \\ &+\tfrac{1}{b}\int_0^t\big(\alpha_2-\alpha(y)\big)\,\kappa\big(z(s)\big)\,y^2(s)\,ds \end{aligned} \tag{19}$$

We define sequences $t_{2k}, t_{1k}, z_{1k}, z_{2k}$ as follows.

$$t_{2k} \equiv \inf\big\{t\in R : z(t)=z_{2k}\big\} \tag{20}$$

$$t_{1k} \equiv \sup\big\{t\in[0,t_{2k}) : z(t)=z_{1k}\big\} \tag{21}$$

$$z_{1k} \equiv 2\pi k+\left(\frac{\pi}{2}\right)\big(1+\operatorname{sgn}(b)\big)-\frac{\pi}{2} \tag{22}$$

$$z_{2k} \equiv 2\pi k+\left(\frac{\pi}{2}\right)\big(1+\operatorname{sgn}(b)\big)+\frac{\pi}{4} \tag{23}$$

We see from these definitions that $z\in[z_{1k},z_{2k}]$ whenever $t\in[t_{1k},t_{2k}]$. To prove the boundedness of z, we first assume unboundedness and show a contradiction. From the unboundedness assumption on z, we know there exists a $k_0>0\in Z$ such that all the above sequences have infinite cardinality.

$$\{z_{1k}\}_{k=k_0}^{\infty}\quad \{z_{2k}\}_{k=k_0}^{\infty}\quad \{t_{1k}\}_{k=k_0}^{\infty}\quad \{t_{2k}\}_{k=k_0}^{\infty}$$

Based on (22) and (23):

$$\cos(z_{1k})=\cos\left(2\pi k+\left(\frac{\pi}{2}\right)\big(1+\operatorname{sgn}(b)\big)-\frac{\pi}{2}\right)$$

$$\cos(z_{2k})=\cos\left(2\pi k+\left(\frac{\pi}{2}\right)\big(1+\operatorname{sgn}(b)\big)+\frac{\pi}{4}\right)$$

If $b<0$:

$$\cos(z_{1k})=\cos\left(2\pi k-\frac{\pi}{2}\right)=0\quad\forall k$$

$$\cos(z_{2k})=\cos\left(2\pi k+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\quad\forall k$$

If $b>0$:

$$\cos(z_{1k})=\cos\left(2\pi k+\frac{\pi}{2}\right)=0\quad\forall k$$

$$\cos(z_{2k})=\cos\left(2\pi k+\frac{5\pi}{4}\right)=\frac{-\sqrt{2}}{2}\quad\forall k$$

From this, the definition of $\kappa(z)$ and the fact that $\beta\in\mathcal{K}_\infty$, we can conclude that when $z\in[z_{1k},z_{2k}]$. This implies that $\frac{1}{b}\kappa(z)$ is less than or equal to zero.

$$\beta\in\mathcal{K}_\infty \Rightarrow \frac{1}{b}\kappa(z)\leq 0\quad \forall z\in[z_{1k},z_{2k}]$$

We know that for all $t\in[0,t_{2k}]$ such that $z(t)\geq z_{1k}$, this relationship also holds.

$$\frac{1}{b}\kappa(z)\leq 0\quad\forall\ \begin{matrix} t\in[0,t_{2k}] \\ z(t)\geq z_{1k}\end{matrix}$$

We can apply the above to the last term of (19) to establish the upper bound when $t=t_{1k}$.

$$\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds$$

We know that $\alpha_2-\alpha\big(y(s)\big)>0\quad\forall s$ and $y^2(s)>0\quad\forall s$ and from our above conclusion

$$\frac{1}{b}\kappa\big(z(t)\big)\leq 0\quad \text{for z (t)} \geq \mathrm{z_{1k}}$$

We know that when $z(t)\geq z_{1k}$, the integral will be negative. Thus, if we look at $z(t)\leq z_{1k}$, we will have an upper bound.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds \\ &\leq \int_{\substack{t\in[0,t_{1k}] \\ z(t)\leq z_{1k}}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds\end{aligned}$$

We can further refine the upper bound by using the fact that

$$\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\leq\frac{\alpha_2-\alpha_1}{|b|}$$

We first substitute the expression for $\kappa(z(s))$.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds \\ &\leq \int_{\substack{t\in[0,t_{1k}] \\ z(t)\leq z_{1k}}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\beta\big(z(s)\big)\cos\big(z(s)\big)\,y^2(s)\,ds\end{aligned}$$

We then bound $\beta\big(z(s)\big)\cos\big(z(s)\big)$ with $\beta\big(z(s)\big)$.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds \\ &\leq \int_{\substack{t\in[0,t_{1k}] \\ z(t)\leq z_{1k}}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\beta\big(z(s)\big)\,y^2(s)\,ds\end{aligned}$$

We bound $\beta\big(z(s)\big)$ with the supremum and factor it out. We also use the bound for $\alpha(y(s))$.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds \\ &\leq \frac{\alpha_2-\alpha_1}{|b|}\sup_{\substack{t\in[0,t_{1k}] \\ z(t)\leq z_{1k}}}\Big\{\beta\big(z(t)\big)\Big\}\int_{\substack{t\in[0,t_{1k}] \\ z(t)\leq z_{1k}}}y^2(s)\,ds\end{aligned}$$

We next solve the expression for z for the integral.

$$z=\frac{1}{2}y^2+\lambda\int_0^t y^2(s)\,ds \Rightarrow \lambda\int_0^t y^2(s)\,ds=z-\frac{1}{2}y^2$$

We substitute this in the bound.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds \\ &\leq \frac{\alpha_2-\alpha_1}{|b|}\sup_{\substack{t\in[0,t_{1k}] \\ z(t)\leq z_{1k}}}\Big\{\beta\big(z(t)\big)\Big\}\left[\frac{z(t)}{\lambda}-\frac{y^2(t)}{2\lambda}\right]_{\substack{t\in[0,t_{1k}] \\ z(t)\leq z_{1k}}}\end{aligned}$$

Since the second term is positive, we can remove it and maintain our upper bound. Plugging in the maximum value of $z(t)$ yields an upper bound.

$$\begin{aligned}&\int_0^{t_{2k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds \\ &\leq \frac{\alpha_2-\alpha_1}{\lambda|b|}\beta(z_{1k})\,z_{1k}\end{aligned} \tag{24}$$

When $t = t_{2k}$, the upper bound remains the same, since when $t > t_{1k}$, $z \geq z_{1k}$ and $\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq 0$, so the integral is negative and we can remove it to create an upper bound similar to the above work.

$$\begin{aligned}&\int_0^{t_{2k}} \tfrac{\alpha_2-\alpha\left(y(s)\right)}{b}\kappa\left(z(s)\right)y^2(s)\,ds\\ \leq &\int_{\substack{t\in[0,t_{1k}]\\ z(t)\leq z_{1k}}} \tfrac{\alpha_2-\alpha\left(y(s)\right)}{b}\kappa\left(z(s)\right)y^2(s)\,ds\end{aligned}$$

We use the same logic to arrive at the upper bound.

$$\begin{aligned}&\int_0^{t_{2k}} \tfrac{\alpha_2-\alpha\left(y(s)\right)}{b}\kappa\left(z(s)\right)y^2(s)\,ds\\ &\leq \tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta(z_{1k})\,z_{1k}\end{aligned} \tag{25}$$

We can now choose $t = t_{2k}$ and apply (25) to (19) to get the following.

$$\begin{aligned}&S(t_{2k}) \leq S(0) + \tfrac{1}{b}\int_0^{z_{2k}}\left(\kappa(s)+b\ell\right)ds\\ &+\tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta(z_{1k})\,z_{1k}\end{aligned} \tag{26}$$

We next examine the second term in the bound in (26).

$$\frac{1}{b}\int_0^{z_{2k}}\left(\kappa(s)+b\ell\right)ds$$

For

$$z \in \left[z_{1k}, z_{2k}-\frac{\pi}{2}\right]$$

When $b < 0$, then we know from before that $\cos(z_{1k}) = 0$ and we can see that:

$$\cos\left(z_{2k}-\frac{\pi}{2}\right) = \cos\left(2\pi k - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \forall k$$

Thus $\kappa\left(z(t)\right) \geq 0$ and:

$$\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq 0$$

When $b > 0$, we know from before that $\cos(z_{1k}) = 0$ and we can see that the following relationship holds.

$$\cos\left(z_{2k}-\frac{\pi}{2}\right) = \cos\left(2\pi k + \frac{3\pi}{4}\right) = \frac{-\sqrt{2}}{2} \quad \forall k$$

Thus $\kappa\left(z(t)\right) \leq 0$ and we can bound $\operatorname{sgn}(b)\,\kappa\left(z(t)\right)$.

$$\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq 0 \quad \forall z \in \left[z_{1k}, z_{2k}-\frac{\pi}{2}\right]$$

For $z \in \left[z_{2k}-\frac{\pi}{2}, z_{2k}\right]$, when $b < 0$, then we know from before that $\cos(z_{2k}) = \frac{\sqrt{2}}{2}$ and from above that the following is true.

$$\cos\left(z_{2k}-\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2} \quad \forall k$$

Thus $\kappa\left(z(t)\right) \geq \left(\frac{1}{\sqrt{2}}\right)\beta\left(z_{2k}-\frac{\pi}{2}\right)$ and we can bound $\operatorname{sgn}(b)\,\kappa\left(z(t)\right)$.

$$\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq -\left(\frac{1}{\sqrt{2}}\right)\beta\left(z_{2k}-\frac{\pi}{2}\right)$$

When $b > 0$, then, again, we know from before that $\cos(z_{2k}) = \frac{-\sqrt{2}}{2}$ and from above that the following is true.

$$\cos\left(z_{2k}-\frac{\pi}{2}\right) = \frac{-\sqrt{2}}{2} \quad \forall k$$

Thus $\kappa\left(z(t)\right) \leq -\left(\frac{1}{\sqrt{2}}\right)\beta\left(z_{2k}-\frac{\pi}{2}\right)$ and we can bound $\operatorname{sgn}(b)\,\kappa\left(z(t)\right)$.

$$\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq -\left(\frac{1}{\sqrt{2}}\right)\beta\left(z_{2k}-\frac{\pi}{2}\right)$$

Using this, we can break up the integral of the term we are interested in. We know that for $z \in \left[z_{1k}, z_{2k}-\frac{\pi}{2}\right]$ the value is negative and can be ignored to create an upper bound and we can easily integrate the $b\ell$ term.

$$\begin{aligned}&\tfrac{1}{b}\int_0^{z_{2k}}\left(\kappa(s)+b\ell\right)ds\\ &\leq \ell z_{2k} + \tfrac{1}{b}\int_0^{z_{1k}}\kappa(s)\,ds + \tfrac{1}{b}\int_{z_{2k}-\pi/2}^{z_{2k}}\kappa(s)\,ds\end{aligned}$$

We can then substitute the above bounds to derive the following upper bound.

$$\begin{aligned}&\tfrac{1}{b}\int_0^{z_{2k}}\left(\kappa(s)+b\ell\right)ds\\ &\leq \ell z_{2k} + \tfrac{1}{|b|}\beta(z_{1k})\,z_{1k} - \tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{2k}-\tfrac{\pi}{2}\right)\end{aligned} \tag{27}$$

We first substitute (27) into (26).

$$\begin{aligned}&S(t_{2k}) \leq S(0) + \ell z_{2k} + \tfrac{1}{|b|}\beta(z_{1k})\,z_{1k} - \tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{2k}-\tfrac{\pi}{2}\right)\\ &+\tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta(z_{1k})\,z_{1k}\end{aligned}$$

We have the following relationship between z_{1k} and z_{2k}.

$$z_{2k} = z_{1k} + 3\pi/4$$

Substituting this, we have the following.

$$\begin{aligned}&S(t_{2k}) \leq S(0) + \ell\left[z_{1k}+3\pi/4\right] + \tfrac{1}{|b|}\beta(z_{1k})\,z_{1k}\\ &-\tfrac{\pi}{2\sqrt{2}|b|}\beta\left(\left[z_{1k}+\tfrac{3\pi}{4}\right]-\tfrac{\pi}{2}\right) + \tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta(z_{1k})\,z_{1k}\end{aligned}$$

We then expand and cancel common terms.

$$\begin{aligned}&S(t_{2k}) \leq S(0) + \ell\tfrac{3\pi}{4} + \ell z_{1k} + \tfrac{1}{|b|}\beta(z_{1k})\,z_{1k}\\ &-\tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{1k}+\tfrac{\pi}{4}\right) + \tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta(z_{1k})\,z\end{aligned}$$

Rearranging we arrive at the final expression.

$$\begin{aligned}&S(t_{2k}) \leq S(0) + \tfrac{3\ell\pi}{4} + \ell z_{1k} + \tfrac{1}{|b|}\left(1+\tfrac{\alpha_2-\alpha_1}{\lambda}\right)\beta(z_{1k})\,z_{1k}\\ &-\tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{1k}+\tfrac{\pi}{4}\right)\end{aligned} \tag{28}$$

We can rewrite (12) as follows, first reversing the sign and then multiplying by z.

$$\begin{aligned}&\lim_{z\to+\infty}\left[\tfrac{\beta(z+\varepsilon)}{z} - c\beta(z)\right] = +\infty\\ &\Rightarrow \lim_{z\to+\infty}\left[c\beta(z) - \tfrac{\beta(z+\varepsilon)}{z}\right] = -\infty\\ &\Rightarrow \lim_{z\to+\infty}\left[c\beta(z)\,z - \beta(z+\varepsilon)\right] = -\infty\end{aligned}$$

We examine (28), repeated below.

$$\begin{aligned}&S(t_{2k}) \leq S(0) + \tfrac{3\ell\pi}{4} + \ell z_{1k} + \tfrac{1}{|b|}\left(1+\tfrac{\alpha_2-\alpha_1}{\lambda}\right)\beta(z_{1k})\,z_{1k}\\ &-\tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{1k}+\tfrac{\pi}{4}\right)\end{aligned}$$

We divide out the factor of $\frac{\pi}{2\sqrt{2}|b|}$. We see that the last two terms approach $-\infty$, forcing the left hand side to be negative. However, we know that $S(t_{2k})$ is positive definite. Therefore, we have a contradiction, and z is thus bounded.

$$\frac{S(t_{2k})}{\frac{\pi}{2\sqrt{2}|b|}} \leq \frac{S(0)}{\frac{\pi}{2\sqrt{2}|b|}} + \frac{\frac{3\ell\pi}{4}}{\frac{\pi}{2\sqrt{2}|b|}} + \frac{\ell z_{1k}}{\frac{\pi}{2\sqrt{2}|b|}} + \frac{\frac{1}{|b|}\left(1+\frac{\alpha_2-\alpha_1}{\lambda}\right)}{\frac{\pi}{2\sqrt{2}|b|}}\beta\left(z_{1k}\right)z_{1k} - \beta\left(z_{1k}+\frac{\pi}{4}\right)$$

Since $z \in \mathcal{L}_\infty$ looking at (11), repeated below, we can conclude that $y \in \mathcal{L}_\infty \cap \mathcal{L}_2$.

$$z = \frac{1}{2}y^2 + \lambda\int_0^t y^2\left(s\right)ds \in \mathcal{L}_\infty \Rightarrow y \in \mathcal{L}_\infty \cap \mathcal{L}_2$$

From (28), we know that $S \in \mathcal{L}_\infty$. Examining the definition of S, we see that $u \in \mathcal{L}_\infty \cap \mathcal{L}_2$.

$$S = w^T\Lambda'\left(y\right)w \in \mathcal{L}_\infty, w = \left[\begin{array}{cc} u & y \end{array}\right]^T \Rightarrow y \in \mathcal{L}_\infty \cap \mathcal{L}_2$$

The boundedness of u and y together with (6) implies that $\dot{u}, \dot{y} \in \mathcal{L}_\infty$.

$$\left\{\begin{array}{c} f\left(y\right) \in \mathcal{L}_\infty \Rightarrow \dot{y} = f\left(y\right) + bu \in \mathcal{L}_\infty \\ u_{nom} \in \mathcal{L}_\infty \Rightarrow \varepsilon\dot{u} = u_{nom} - u \in \mathcal{L}_\infty \end{array}\right\}$$

Together, this allows us to apply Barbalat's Lemma and conclude that $y(t), u(t)$ go to zero.

$$\lim_{t\to\infty} y\left(t\right) = \lim_{t\to\infty} u\left(t\right) = 0$$

And finally, given the definition of u_{nom}, we see that u_{nom} also goes to zero.

$$\begin{array}{l} \lim\limits_{t\to\infty} y\left(t\right) = 0, \lim\limits_{t\to\infty} z\left(t\right) = 0 \\ \Rightarrow \lim\limits_{t\to\infty} u_{nom} = \lim\limits_{t\to\infty} \kappa\left(z\left(t\right)\right)y\left(t\right) = 0 \end{array}$$

This completes the proof of Theorem 1, showing the all signals are bounded and converge to zero. ∎

Corollary 1 Let the closed-loop system described by the linear system with ignored fast actuator dynamics

$$\left\{\begin{array}{c} \dot{y} = \alpha y + bu \\ \varepsilon\dot{u} = u_{nom} - u \end{array}\right\} \tag{29}$$

and controller (10), (11).

$$u_{nom} = \kappa\left(z\right)y$$

$$z = \frac{1}{2}y^2 + \lambda\int_0^t y^2\left(s\right)ds$$

If $\varepsilon\left(\lambda + \alpha_2\right) < 1$, and $\kappa(\cdot)$ is a Nussbaum function then, all closed-loop signals are bounded and

$$\lim_{t\to\infty} y\left(t\right) = \lim_{t\to\infty} u\left(t\right) = \lim_{t\to\infty} u_{nom}\left(t\right) = 0.$$

Proof: In the case of a linear system, we have the following.

$$\alpha\left(y\right) = \alpha_1 = \alpha_2 = \alpha$$

In this case, the last terms in (19) and (26) will cancel and they will become the following.

$$S - \int_0^t w^T\left(s\right)\Lambda\left(y\right)w\left(s\right)ds \leq S\left(0\right) + \ell z\left(t\right) + \frac{1}{b}\int_0^{z(t)}\left(\kappa\left(s\right)\right)ds$$

$$S\left(t_{2k}\right) \leq S\left(0\right) + \frac{1}{b}\int_0^{z_{2k}}\left(\kappa\left(s\right) + b\ell\right)ds$$

The derivation of (24) and (25) is no longer required. The proof otherwise proceeds the same. ∎

III. Simulation examples

A. Linear system

A simulation was performed on the linear system with ignored actuator dynamics described in (29).

$$\left\{\begin{array}{c} \dot{y} = \alpha y + bu \\ \varepsilon\dot{u} = u_{nom} - u \end{array}\right\}$$

$$\alpha = 0.8 \quad b = 0.05 \quad \varepsilon = 0.1 \quad y\left(0\right) = 5 \quad u\left(0\right) = 0$$

Three systems were compared: a Nussbaum gain controller described in (30), a nonlinear PI controller (10) and (11) with a non-Nussbaum gain ($\kappa\left(z\right) = z\cos\left(z\right)$), and a nonlinear PI controller with a Nussbaum gain ($\kappa\left(z\right) = z^2\cos\left(z\right)$).

Nussbaum Gain Controller:

$$\left\{\begin{array}{c} u_{nom} = \zeta^2\cos\left(\zeta\right)y \\ \dot{\zeta} = \lambda y^2 \end{array}\right\} \quad \begin{array}{c} \zeta\left(0\right) = 0 \\ \lambda = 0.15 \end{array} \tag{30}$$

As seen in Figure 4 of [1], only the nonlinear PI controller with a Nussbaum gain provides convergent solutions.

PI

$$u_{\text{nom}} = \kappa\left(z\right)y \quad z = \tfrac{1}{2}y^2 + \lambda\int_0^t y^2\left(s\right)ds$$

$$\begin{array}{cc} \kappa\left(z\right) = z\cos\left(z\right) & \text{Not Nussbaum} \\ \kappa\left(z\right) = z^2\cos\left(z\right) & \text{Nussbaum} \end{array}$$

The results of the author's simulations were replicated using the Simulink models seen in Figures 1, 2, and 3. A graph of the results can be seen in Figure 4, which matches the results achieved by the author.

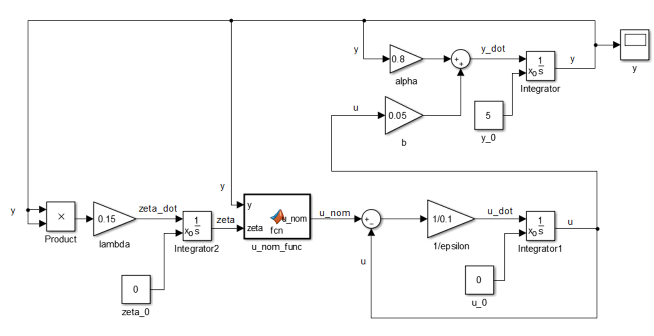


Fig. 1: Simulink Model for LSIAD + NG

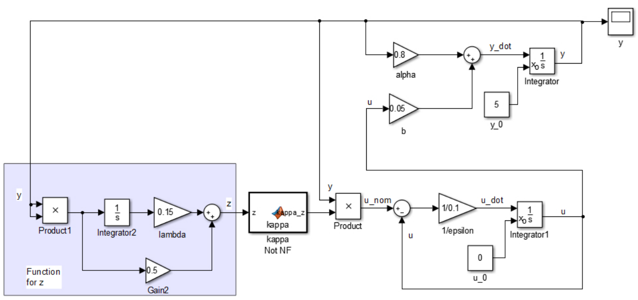


Fig. 2: Simulink Model for LSIAD + nPI

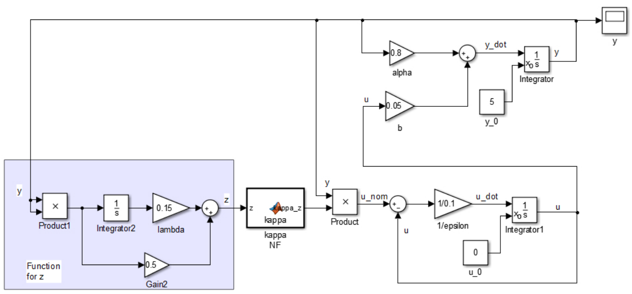


Fig. 3: Simulink Model for LSIAD + nPI-N

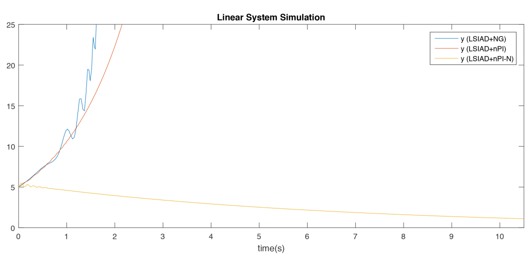


Fig. 4: Results for Linear System

B. Nonlinear system

A simulation of the nonlinear system (6) was also performed with parameters conforming the assumptions of Theorem 1. As seen in Figure 5 of [1], $y(t)$, $u(t)$, and $u_{nom}(t)$ are all bounded and converge to zero as expected.

$$\left\{ \begin{array}{c} \dot{y} = f(y) + bu \\ \varepsilon \dot{u} = u_{nom} - u \end{array} \right\}$$

$$f(x) = 3\left[1 + 2\sin\left(\exp(x)\right)\right] x$$

$$b = 1 \quad \alpha_1 = -3 \quad \alpha_2 = 9 \quad \lambda = 0.5 \quad \varepsilon < \frac{1}{\alpha_2 + \lambda} = 0.105$$

$$u = \kappa(z)\, y \quad z = \frac{1}{2} y^2 + \lambda \int_0^t y^2(s)\, ds$$

$$\kappa(z) = \left[\exp\left(\frac{z^2}{10}\right) - 1\right] \cos(z) \qquad \begin{array}{c} \varepsilon = 0.1 \\ u(0) = 0 \\ y(0) = 5 \end{array}$$

An attempt was made to replicate the author's simulation using the authors settings with the Simulink model seen in Figure 5. However, the results, seen in Figure 6a, did not match the author's. y, u and u_{nom} did go to zero, but u and u_{nom} grew unreasonably large at the start. Based on the author's previous work in [2], a slight adjustment was made so $\lambda = 2.5$ and $y(0) = 4$. The results, seen in Figure 6b are more consistent with the desired outcome, such that y, u and u_{nom} go to zero and remain reasonable throughout. The large amount of oscillation at the beginning suggested the need for a larger value of λ which acts as a dampening factor and a singularity near $y = 4.6$, required a shifting of the initial condition for y.

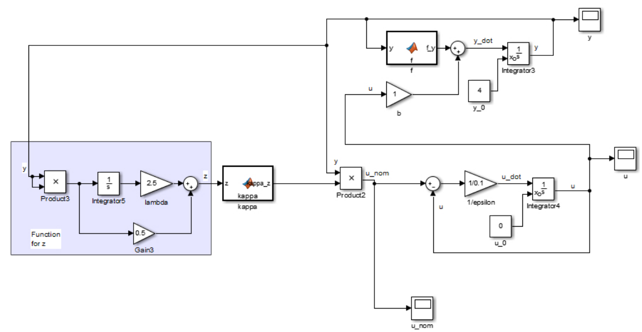
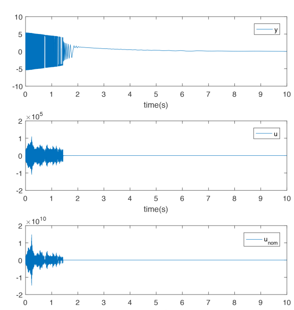
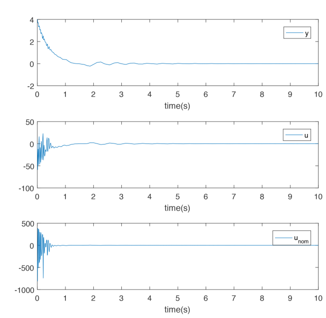


Fig. 5: Simulink Model for the Nonlinear System



(a) Results for the Nonlinear System Using Given Parameters



(b) Results for the Nonlinear System Using Modified Parameters

Fig. 6: Nonlinear System

IV. Conclusions

As seen in Theorem 1 and the accompanying simulations, using a nonlinear PI controller with an appropriately selected Nussbaum gain will allow for robust control of sector bounded nonlinear systems with unknown control directions.

References

[1] Psillakis, Haris E. "Robustness of the nonlinear PI control method to ignored actuator dynamics." arXiv preprint arXiv:1408.3229 (2014).

[2] Psillakis HE. An extension of the Georgiou-Smith example: boundedness and attractivity in the presence of unmodelled dynamics via nonlinear PI control,(e-print, arXiv:1407.7213 [cs.SY]).